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Uniqueness of the $(22, 891, 1/4)$ spherical code

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Abstract. We use techniques of Bannai and Sloane to give a new proof that there is a unique $(22, 891, 1/4)$ spherical code; this result is implicit in a recent paper by Cuypers. We also correct a minor error in the uniqueness proof given by Bannai and Sloane for the $(23, 4600, 1/3)$ spherical code.

Contents

1. Introduction

An (n, N, t) spherical code is a set of N points on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that no two distinct points in the set have inner product greater than t. In other words, the angles between them are all at least $\cos^{-1} t$. The fundamental problem is to maximize N for a given value of t, or equivalently to minimize t given N. Of course, for specific values of N and t, maximality of N given t is not equivalent to minimality of t given N, but complete solutions of these problems for all parameter values would be equivalent.

Linear programming bounds are a powerful tool for proving upper bounds on N given t (see [DGS], [KL], or Chapter 9 in [CS]). In particular, they prove sharp bounds in a number of important special cases, listed in [Lev]. Once a code has been proved optimal, it is natural to ask whether it is unique up to orthogonal transformations. That is known in every case to which linear programming bounds apply except for one infinite family that is not always unique (see Appendix A of [CK] for an overview). However, one should not expect uniqueness to hold in general for optimal spherical codes: for example, the D_5 kissing arrangement appears to be

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an optimal 40-point spherical code in $\mathbb{R}^5$, but there is at least one other $(5, 40, 1/2)$ code (see [Lee]).

One noteworthy case is the $(22, 891, 1/4)$ code. A proof of uniqueness is implicit in the recent paper [C] by Cuypers, but we are unaware of any explicit discussion of uniqueness in the literature (by contrast, every other case has been explicitly analyzed). In this paper, we apply techniques from [BS] to give a new proof that it is unique.

This code arises naturally in the study of the Leech lattice in $\mathbb{R}^{24}$ (see [E] or [CS] for background). In the sphere packing derived from the Leech lattice, each sphere is tangent to 196560 others. The points of tangency form a $(24, 196560, 1/2)$ code known as the kissing configuration of the Leech lattice. It can be viewed as a packing in 23-dimensional spherical geometry, whose kissing configuration is a $(23, 4600, 1/3)$ code. The $(22, 891, 1/4)$ code is obtained by taking the kissing configuration once more; it is well defined because the automorphism group of the Leech lattice acts distance transitively on the $(24, 196560, 1/2)$ code. All three of these codes are optimal (in fact, universally optimal—see [CK]), although that is not known for the $(21, 336, 1/5)$ code that comes next in the sequence. The linear programming bounds are not sharp for the $(21, 336, 1/5)$ code, and we make no conjecture as to whether it is optimal. Its kissing configuration is a $(20, 170, 1/6)$ code whose symmetry group does not even act transitively: there are two orbits of points, one with 10 points (forming the midpoints of the edges of a regular 4-dimensional simplex) and one with 160 points. Because of the lack of transitivity, this configuration has two different types of kissing configurations, and it seems fruitless to continue examining iterated kissing configurations. The $(20, 170, 1/6)$ code is not universally optimal and probably not even optimal.

One can also construct the $(22, 891, 1/4)$ code using a 6-dimensional Hermitian space over $\mathbb{F}_4$. Points in the configuration correspond to 3-dimensional totally isotropic subspaces, with the inner product between two points ($-1/2$, $1/4$, or $-1/8$) determined by the dimension of the intersection of the corresponding subspaces (2, 1, or 0, respectively). The graph with these subspaces as vertices and with edges between pairs of subspaces with intersection dimension 2 is the dual polar graph associated with the group $\mathrm{PSU}(6, 2)$ (see Section 9.4 in [BCN]). In the paper [C], it is implicit in the proof of Proposition 2.2 that a $(22, 891, 1/4)$ spherical code must have the combinatorial structure of a $(2, 4, 20)$ regular near hexagon, which is equivalent to this dual polar space structure (see [SY]). Uniqueness then follows from the classification of all polar spaces of rank at least 3 by Tits in [T]. By contrast, our proof makes use of entirely different machinery.

The linear programming bounds not only prove bounds on spherical codes, but also provide additional information about the codes that achieve a given bound. When used with the auxiliary polynomial $(x+1/2)^2(x+1/8)^2(x-1/4)$, they prove that every code in S^{21} with maximal inner product $1/4$ has size at most 891, and that equality is achieved iff all inner products between distinct vectors are in $\{-1/2, -1/8, 1/4\}$ and the code is a spherical 5-design. Recall that a spherical t-design is a finite subset of the sphere $S^{n-1} \subset \mathbb{R}^n$ such that for every polynomial function $p\colon \mathbb{R}^n \to \mathbb{R}$ of total degree at most t, the average of p over the design equals its average over the entire sphere.

The techniques we use to prove uniqueness were developed by Bannai and Sloane in [BS], and we follow their approach quite closely. (Note that their paper is

reprinted as Chapter 14 of [CS].) They proved uniqueness for the $(24, 196560, 1/2)$ and $(23, 4600, 1/3)$ codes, as well as analogous codes derived from the E_8 root lattice. Here we correct a minor error in their proof for the $(23, 4600, 1/3)$ code. They construct a lattice L and conclude their proof by saying "and hence L must be the Leech lattice," but in fact it is not the Leech lattice (it is a sublattice of index 2). At the end of this paper we explain the problem and how to correct it.

One small difference between this paper and [BS] is that the $(22, 891, 1/4)$ code is not a tight spherical design, whereas all the designs dealt with in [BS] are tight. (A tight spherical $(2e+1)$-design in $\mathbb{R}^n$ is one with $2\binom{n+e-1}{n-1}$ points, which by Theorem 5.12 of [DGS] is a lower bound for the number of points.) However, no fundamental changes in the techniques are needed. The only important difference is that we cannot conclude that the $(22, 891, 1/4)$ code is the only 891-point spherical 5-design in $\mathbb{R}^{22}$, as we could if it were tight.

2. Uniqueness of the $(22, 891, 1/4)$ code

Theorem 1. *There is a unique* $(22, 891, 1/4)$ *spherical code, up to orthogonal transformations of* $\mathbb{R}^{22}$.

Let $\mathcal{C}$ be such a code. We begin with the observation that by the sharpness of the linear programming bounds, $-1/2$, $-1/8$, and $1/4$ are the only possible inner products that can occur between distinct points in $\mathcal{C}$. Let $u_1, \dots, u_{891}$ be the points in $\mathcal{C}$, and let

$$\begin{aligned} U_i &= (1, 1/\sqrt{3}, \sqrt{8/3}\, u_i), \\ V_0 &= (2, 0, \dots, 0), \text{ and} \\ V_1 &= (1, \sqrt{3}, 0, \dots, 0) \end{aligned}$$

be vectors in $\mathbb{R}^{24}$. The slightly nonstandard notation $(1, 1/\sqrt{3}, \sqrt{8/3}\, u_i)$ of course means the concatenation of the vectors $(1, 1/\sqrt{3})$ and $\sqrt{8/3}\, u_i$.

It is easy to check that all these vectors have norm 4 and the inner product between any two of them is an integer; specifically, $\langle U_i, U_j \rangle$ is 4, 2, 1, or 0 according as $\langle u_i, u_j \rangle$ is 1, 1/4, $-1/8$, or $-1/2$, respectively. Let L be the lattice spanned by $U_1, \dots, U_{891}$, V_0, and V_1. It follows that L is an even lattice (i.e., all vectors have even norms). We will show that L is uniquely determined, up to orthogonal transformations of $\mathbb{R}^{24}$ that fix V_0 and V_1, as is $\{U_1, \dots, U_{891}\}$.

In what follows, vectors in $\mathbb{R}^{24}$ are generally denoted by uppercase letters and vectors in $\mathbb{R}^{22}$ by lowercase letters. One exception is the standard basis $e_1, \dots, e_{24}$ of $\mathbb{R}^{24}$.

Lemma 2. *The minimal norm* $\langle V, V \rangle$ *for* $V \in L \setminus \{0\}$ *is* 4.

Proof. Suppose there exists $V \in L$ with $\langle V, V \rangle = 2$. Then $\langle V, W \rangle \in \{0, \pm 1, \pm 2\}$ for all $W \in \{V_0, V_1, U_1, \dots, U_{891}\}$, because $\langle V, W \rangle \in \mathbb{Z}$ and $|\langle V, W \rangle| \le |V||W| = 2\sqrt{2}$.

Now let $V = (x, y/\sqrt{3}, \sqrt{8/3}\, u)$ with $u \in \mathbb{R}^{22}$ and $x, y \in \mathbb{R}$. We note that x and y must be integers of the same parity, from the description of the generators of the lattice L. Also, we must have $x^2 + y^2/3 \le 2$, by the condition on the norm of V. This implies that $(x, y) \in \{(0,0), (0, \pm 2), (\pm 1, \pm 1)\}$. We can furthermore assume that $(x, y) \in \{(0,0), (0,2), (1, \pm 1)\}$, because otherwise we replace V with $-V$. If $(x, y) = (0, 2)$, then $\langle V, V_1 \rangle = 2$ and thus $|V_1 - V|^2 = 2$, so we can replace V with

$V_1 - V$ and (x, y) with $(1, 1)$. If $(x, y) = (1, -1)$, then we can replace V with $V_0 - V$ and (x, y) with $(1, 1)$. We can therefore assume that (x, y) is $(0, 0)$ or $(1, 1)$.

If $(x, y) = (1, 1)$, then we claim that there exists an i such that $\langle V, U_i \rangle = 2$, in which case replacing V with $V - U_i$ reduces to the case of $(x, y) = (0, 0)$. To prove the existence of such an i, consider the point $u \in \mathbb{R}^{22}$, which has $|u| = 1/2$. For each i, if $\langle V, U_i \rangle \in \{-2, -1, 0, 1\}$, then $\langle u, u_i \rangle \in \{-5/4, -7/8, -1/2, -1/8\}$. If that were always the case, then the set $\{2u, u_1, \dots, u_{891}\}$ would be a $(22, 892, 1/4)$ spherical code, which is impossible.

We are left with only one case, namely that $(x, y) = (0, 0)$. Then $|u| = \sqrt{3}/2$. The inner products $\langle u, u_i \rangle$ must lie in the set $\{0, \pm 3/8, \pm 3/4\}$, corresponding to the restriction that $\langle V, U_i \rangle \in \{0, \pm 1, \pm 2\}$. Let $N_0, N_{3/8}, N_{-3/8}, N_{3/4}, N_{-3/4}$ be the numbers of vectors u_i that have inner products $0, 3/8, -3/8, 3/4, -3/4$, respectively, with u. Now from the fact that $\mathcal{C}$ is a 5-design (which we obtain from the linear programming bounds), we observe that for every polynomial $p(x)$ of degree at most 5,

$$\frac{\sum_{\alpha \in \{0, 3/8, -3/8, 3/4, -3/4\}} N_\alpha p(\alpha)}{891} = \int_{S^{21}} p\big(\langle z, u \rangle\big)\, d\mu(z),$$

where the surface measure μ on S^{21} has been normalized to have total volume 1.

The right side does not depend on the direction of u, only on its magnitude, and it is easily evaluated when $p(x) = x^i$: for i odd it vanishes, and for i even it equals

$$|u|^i \frac{i!(22/2 - 1)!}{(i/2 + 22/2 - 1)!(i/2)! 2^i} = \frac{i!\, 10!}{(10 + i/2)!(i/2)!} \left(\frac{\sqrt{3}}{4}\right)^i.$$

We write down five equations corresponding to the monomials $p(x) = 1$, x, x^2, x^3, and x^4 and solve the resulting system of equations to get

$$(N_0, N_{3/8}, N_{-3/8}, N_{3/4}, N_{-3/4}) = (657, 120, 120, -3, -3).$$

The negative numbers give us the contradiction. □

As an immediate corollary we observe that the (integral) inner product between two minimal vectors of L cannot be ± 3 and so must lie in $\{0, \pm 1, \pm 2, \pm 4\}$: if $\langle U, V \rangle = 3$ with U and V minimal vectors, then $|U - V|^2 = |U|^2 + |V|^2 - 2\langle U, V \rangle = 2$, contradicting Lemma 2.

It follows from Theorem 7.4 in [DGS] that because $\mathcal{C}$ is a 5-design in which 3 inner products other than 1 occur and $5 \geq 2 \cdot 3 - 2$, the points in $\mathcal{C}$ form a 3-class association scheme when pairs of points are grouped according to their inner products. In other words, given $\alpha, \beta, \gamma \in \{-1/2, -1/4, 1/8, 1\}$, there is a number $P_\gamma(\alpha, \beta)$ such that whenever $\langle u_i, u_j \rangle = \gamma$, there are exactly $P_\gamma(\alpha, \beta)$ points u_k such that $\langle u_i, u_k \rangle = \alpha$ and $\langle u_j, u_k \rangle = \beta$. These numbers are called intersection numbers and are determined in the proof of the theorem in [DGS]. We have tabulated them in Table 1 (note that $P_\gamma(\alpha, \beta) = P_\gamma(\beta, \alpha)$, $P_\gamma(\alpha, 1)$ is the Kronecker delta function $\delta_{\alpha,\gamma}$, and $P_1(\alpha, \beta) = 0$ unless $\alpha = \beta$).

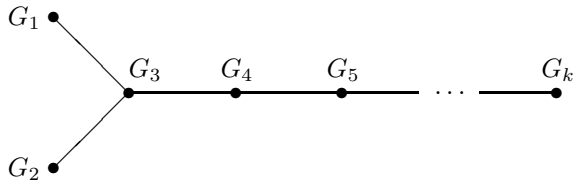
Lemma 3. *The lattice L contains a sublattice isometric to $\sqrt{2}D_{24}$ and containing V_0 and V_1.*

Recall that the minimal norm in D_n is 2, so it is 4 in $\sqrt{2}D_n$.

$P_1(1/4,1/4) = 336$	$P_1(-1/8,-1/8) = 512$	$P_1(-1/2,-1/2) = 42$
$P_{1/4}(1/4,1/4) = 170$	$P_{1/4}(-1/8,-1/8) = 320$	$P_{1/4}(-1/2,-1/2) = 5$
$P_{1/4}(1/4,-1/8) = 160$	$P_{1/4}(1/4,-1/2) = 5$	$P_{1/4}(-1/8,-1/2) = 32$
$P_{-1/8}(1/4,1/4) = 105$	$P_{-1/8}(-1/8,-1/8) = 280$	$P_{-1/8}(-1/2,-1/2) = 0$
$P_{-1/8}(1/4,-1/8) = 210$	$P_{-1/8}(1/4,-1/2) = 21$	$P_{-1/8}(-1/8,-1/2) = 21$
$P_{-1/2}(1/4,1/4) = 40$	$P_{-1/2}(-1/8,-1/8) = 256$	$P_{-1/2}(-1/2,-1/2) = 1$
$P_{-1/2}(1/4,-1/8) = 256$	$P_{-1/2}(1/4,-1/2) = 40$	$P_{-1/2}(-1/8,-1/2) = 0$

TABLE 1. Intersection numbers for a $(22, 891, 1/4)$ code.

Proof. We prove by induction on n that there exist minimal vectors $G_1, \dots, G_n \in L$ such that $\langle G_1, G_2\rangle = 0$, $\langle G_1, G_3\rangle = -2$, $\langle G_i, G_{i+1}\rangle = -2$ for $2 \le i \le n-1$, and all other inner products vanish. In other words, for $3 \le k \le n$, the vectors $G_1, \dots, G_k$ span a copy of $\sqrt{2}D_k$, as one can see from the Dynkin diagram of D_k:



In what follows we refer to this copy as $\sqrt{2}D_k$, and we write $G_1 = -\sqrt{2}(E_1 + E_2)$ and $G_i = \sqrt{2}(E_{i-1} - E_i)$ for $i \ge 2$, so $E_1, \dots, E_n$ is an orthonormal basis of the ambient space $\mathbb{R}D_n = \mathbb{R} \otimes_{\mathbb{Z}} \sqrt{2}D_n$ of $\sqrt{2}D_n$.

We will furthermore choose this sublattice to contain V_0 and V_1 when $n \ge 5$.

For $n = 3$, the existence of such vectors follows immediately from the fact that the intersection numbers $P_1(-1/2,-1/2) = 42$ and $P_{-1/2}(1/4,1/4) = 40$ are positive. Choose $G_1 = U_i$ for any i. Then among $U_1, \dots, U_{891}$ there are 42 choices for G_2, and 40 choices among $-U_1, \dots, -U_{891}$ for G_3 given G_2.

Now suppose the assertion holds up to dimension n, with $3 \le n < 24$. As a first step we show that there are at least 43 minimal vectors W in L such that $\langle G_i, W\rangle = 2$ for $i \in \{1, 2\}$, whereas in $\sqrt{2}D_n$ there are only $2n - 4 \le 42$, namely $G_1 + G_2 + \cdots + G_k$ and $-G_3 - G_4 - \cdots - G_k$ with $3 \le k \le n$. (Checking this assertion for $\sqrt{2}D_n$ is a straightforward exercise in manipulating coordinates.) The next lattice $\sqrt{2}D_{n+1}$ will be spanned by $\sqrt{2}D_n$ and such a vector W.

To construct these vectors W we work as follows. Renumbering the vectors if necessary, we can assume that $U_1, \dots, U_{40}$ satisfy $\langle G_i, U_j\rangle = 2$ for $i \in \{1, 2\}$ and $1 \le j \le 40$ (because $P_{-1/2}(1/4,1/4) = 40$). The vectors V_0 and V_1 also satisfy $\langle G_i, V_j\rangle = 2$ for $i \in \{1, 2\}$ and $j \in \{0, 1\}$. We must still find one more choice of W. To do so, note that $P_{-1/2}(-1/2,-1/2) = 1$. Hence there is a unique vector U_ℓ such that $\langle U_\ell, G_1\rangle = \langle U_\ell, G_2\rangle = 0$. The vector $V_2 = V_0 - U_\ell$ is another choice for W (we could also choose $V_1 - U_\ell$, but we will not require that many possibilities).

The 43 vectors $U_1, \dots, U_{40}, V_0, V_1, V_2$ are all distinct: the only possible danger is if V_2 equals one of the other vectors. Because $V_2 = V_0 - U_\ell$, clearly $V_2 \neq V_0$, and $V_2 \neq V_1$ follows from looking at the second coordinate in the definitions of V_0, V_1, U_i. Similarly, $V_2 = U_i$ is impossible because comparing second coordinates shows that $V_0 \neq U_i + U_\ell$.

Thus, there are at least 43 minimal vectors W satisfying $\langle G_k, W\rangle = 2$ for $k = 1, 2$, whereas in $\sqrt{2}D_n$ there are at most 42. Choose a minimal vector W with this property such that $W \notin \sqrt{2}D_n$, and in particular choose $W = V_0$ or $W = V_1$ if possible. (That will ensure that $V_0, V_1 \in \sqrt{2}D_n$ if $n \geq 5$.)

This vector W cannot be in $\mathbb{R}D_n$: if $W = \sum_{i=1}^n c_i E_i$, then $\langle G_k, W\rangle = 2$ for $k \in \{1, 2\}$ implies $c_1 = 0$ and $c_2 = -\sqrt{2}$. For $3 \leq i \leq n$, $\sqrt{2}(E_2 \pm E_i)$ is a minimal vector in $\sqrt{2}D_n \subset L$, and therefore $\langle W, \sqrt{2}(E_2 \pm E_i)\rangle \in \{0, \pm 1, \pm 2\}$. (The inner product cannot be ± 4 because $\sqrt{2}(E_2 \pm E_i) \in \sqrt{2}D_n$ but $W \notin \sqrt{2}D_n$.) Because $\langle W, \sqrt{2}(E_2 \pm E_i)\rangle = -2 \pm \sqrt{2}c_i$, it follows that $c_3 = c_4 = \cdots = c_n = 0$, which contradicts $\langle W, W\rangle = 4$.

Choose E_{n+1} so that $\{E_1, \dots, E_{n+1}\}$ is an orthonormal basis for $\mathbb{R}D_n \oplus \mathbb{R}W$, and let $W = c_1 E_1 + \cdots + c_{n+1}E_{n+1}$. Then the same calculation gives $c_1 = 0$, $c_2 = -\sqrt{2}$, $c_3 = \cdots = c_n = 0$, and $c_{n+1} = \pm\sqrt{2}$. Thus, $\sqrt{2}D_n$ and W span a copy of $\sqrt{2}D_{n+1}$ contained in L. □

It will be convenient in the rest of the proof of Theorem 1 to change coordinates to agree with the standard coordinates for the Leech lattice (see [CS, p. 131]). To do so, choose coordinates so that L contains the usual lattice $\sqrt{2}D_{24}$ (i.e., $\sqrt{2}$ times all the integral vectors with even coordinate sum), with $V_0, V_1 \in \sqrt{2}D_{24}$. We can furthermore assume that $V_0 = (4, 4, 0, \dots, 0)/\sqrt{8}$ and $V_1 = (4, 0, 4, 0, 0, \dots, 0)/\sqrt{8}$, because the automorphism group of $\sqrt{2}D_{24}$ acts transitively on pairs of minimal vectors with inner product 2. (We write the vectors in this way, with $4/\sqrt{8}$ instead of $\sqrt{2}$, because it will prove helpful in dealing with the Leech lattice.)

In these new coordinates, all inner products are of course preserved, but the coordinates for $U_1, \dots, U_{891}$ are no longer the same as those we previously used. Let $e_1, \dots, e_{24}$ denote the standard basis of $\mathbb{R}^{24}$ with respect to the new coordinates. From this point on, all uses of coordinates refer to the new coordinates.

We wish to show that the vectors $U_1, \dots, U_{891}$ are uniquely determined, up to orthogonal transformations of $\mathbb{R}^{24}$ fixing V_0 and V_1, which include of course permutations and sign changes of the last 21 coordinates. Let $W = (w_1, \dots, w_{24})/\sqrt{8}$ be one of the U_i's. Then

$$\sum_{i=1}^{24} w_i^2 = 8|W|^2 = 32,$$

$$(w_i \pm w_j)/2 = \big\langle W, \sqrt{2}(e_i \pm e_j)\big\rangle \in \{0, \pm 1, \pm 2, \pm 4\}$$

for $i \neq j$,

$$(w_1 + w_2)/2 = \langle W, V_0\rangle = 2,$$

and

$$(w_1 + w_3)/2 = \langle W, V_1\rangle = 2.$$

From the above conditions we see that each w_i is an integer (because $(w_i + w_j)/2$ and $(w_i - w_j)/2$ are), and that they are all at most 4 in absolute value and of the

same parity. A little more work shows that the only possibilities are

$$\sqrt{8}W = \begin{cases} 4(e_1 \pm e_j) & \text{with } j \geq 4, \\ 4(e_2 + e_3), & \\ 2(e_1 + e_2 + e_3) + 2\sum_{k=1}^{5} \pm e_{j_k} & \text{with } 3 < j_1 < j_2 < \cdots < j_5, \text{ or} \\ 3e_1 + e_2 + e_3 + \sum_{j=4}^{24} \pm e_j. & \end{cases} \tag{1}$$

To prove this, note first that $w_1 \geq 0$ since $(w_1 + w_2)/2 = 2$ and $w_2 \leq 4$. If $w_1 = 0$, then $w_2 = w_3 = 4$, and $w_i = 0$ for $i > 3$ because $|W|^2 = 4$. If $w_1 = 1$, then $w_2 = w_3 = 3$ and hence $(w_2 + w_3)/2 = 3$, which is impossible. If $w_1 = 2$, then $w_2 = w_3 = 2$; the constraint that $(w_1 \pm w_i)/2 \in \{0, \pm 1, \pm 2, \pm 4\}$ rules out $w_i = \pm 4$, so all remaining coordinates are in $\{0, \pm 2\}$, and there must be five more ± 2's because $|W|^2 = 4$. If $w_1 = 3$, then $w_2 = w_3 = 1$, and $(w_1 \pm w_i)/2 \in \{0, \pm 1, \pm 2, \pm 4\}$ rules out $w_i = \pm 3$ for $i > 1$, so all remaining coordinates must be ± 1. Finally, if $w_1 = 4$, then $w_2 = w_3 = 0$, and $(w_1 \pm w_i)/2 \in \{0, \pm 1, \pm 2, \pm 4\}$ implies $w_i \in \{0, \pm 4\}$; exactly one more coordinate must be ± 4 because $|W|^2 = 4$.

Call the cases enumerated in Equation (1) above Case I, Case II, Case III, and Case IV, respectively.

By abuse of notation, view $\{0,1\}^{21}$ as being contained in $\mathbb{Z}^{21} = \sum_{i=4}^{24} \mathbb{Z}e_i$. We define a code $\mathcal{D} \subset \{0,1\}^{21}$ by stipulating that $c \in \mathcal{D}$ iff $(2(e_1+e_2+e_3)+2c+4z)/\sqrt{8}$ is one of the U_i's for some $z \in \mathbb{Z}^{21}$. This corresponds to Case III above. The codewords in $\mathcal{D}$ have weight 5, and the minimum distance between codewords is at least 8, since the minimum distance between vectors of the lattice L is 2. (If $(2(e_1 + e_2 + e_3) + 2c_1 + 4z_1)/\sqrt{8}$ and $(2(e_1 + e_2 + e_3) + 2c_2 + 4z_2)/\sqrt{8}$ are both as above, then $(2(c_1 - c_2) + 4(z_1 - z_2))/\sqrt{8} \in L$. One can add an element of $\sqrt{2}D_{24}$ to cancel all of $4(z_1 - z_2)/\sqrt{8}$ except for one coordinate, and another to cancel the remaining coordinate at the cost of changing the sign of one of the ± 2's occurring in $2(c_1 - c_2)/\sqrt{8}$. Then if the distance between the codewords c_1 and c_2 in $\mathcal{D}$ is less than 8, the resulting vector in L has length less than 2.)

It follows from the linear programming bounds for constant-weight binary codes (see [MS, p. 545]) that the largest such code has size 21. In particular, it is a projective plane over $\mathbb{F}_4$ (the points are coordinates and the lines are the supports of the codewords), or equivalently an $S(2, 5, 21)$ Steiner system, and it is thus unique up to permutations of the coordinates (Satz 1 in [W]). Also, for each codeword of $\mathcal{D}$, we can only use at most half of the possible sign assignments in the ± 2's in Case III, since otherwise we would get two elements of L that agree except for one sign and are thus at distance $(2 - (-2))/\sqrt{8} = \sqrt{2}$, which is again a contradiction. This gives a total of at most $2^4 \cdot 21 = 336$ possible minimal vectors for Case III.

Similarly, for Case IV, define a code $\mathcal{E} \subset \{0,1\}^{21}$ so that $c \in \mathcal{E}$ iff

$$\left(3e_1 + e_2 + e_3 + 2c - \sum_{i=4}^{24} e_i\right)/\sqrt{8}$$

is one of the U_i's. We note as before that codewords have distance at least 8 from each other, and also at most 16 (otherwise two U_i's would be too far apart). The largest such code has 512 codewords, as is easily proved using linear programming bounds (see Theorem 20 of Chapter 17 in [MS, p. 542]), if one takes into account both the minimal and the maximal distance. This is more subtle than it might at first appear, because the linear programming bounds are not in fact sharp if one

uses only the minimal distance. We conclude that there are at most 512 vectors in Case IV.

In all, the number of possible U_i's is at most $2 \cdot 21 + 1 + 336 + 512 = 891$. On the other hand, we already know that there are 891 of them. This forces the codes $\mathcal{D}$ and $\mathcal{E}$ to have the greatest possible size. In particular, $\mathcal{D}$ is uniquely determined, up to permutation of coordinates. These coordinate permutations are orthogonal transformations of $\mathbb{R}^{24}$ that fix V_0 and V_1 and preserve $\sqrt{2}D_{24}$. To complete the proof of uniqueness, it will be enough to show that after performing further such transformations that preserve the code $\mathcal{D}$, we can specify all the vectors of Cases III and IV exactly.

Let $W_0 = \left(3e_1 + e_2 + e_3 + 2c_0 - \sum_{i=4}^{24} e_i\right)/\sqrt{8}$ be a fixed vector from Case IV. Let $i_1, \dots, i_r$ be the places (between 4 and 24) where c_0 has a 1. Then let ϕ be the composition of reflections in the corresponding hyperplanes (i.e., change the signs of those coordinates). Applying ϕ clearly fixes V_0 and V_1 and it takes the vector W_0 to $\left(3e_1 + e_2 + e_3 - \sum_{i=4}^{24} e_i\right)/\sqrt{8}$. It also preserves the code $\mathcal{D}$. Thus, we can assume that $W_0 = \left(3e_1 + e_2 + e_3 - \sum_{i=4}^{24} e_i\right)/\sqrt{8}$. Now we try to determine the precise form of the vectors of Case III. We know that they have ± 2 entries in the positions of the code $\mathcal{D}$; the only question is if we can pin down the positions of the signs. Let $d \in \mathcal{D}$ be a codeword, and let V be any vector in Case III with ± 2's at the positions specified by the codeword d. Suppose r of these are -2's and $5 - r$ are 2's. Then taking the inner product with W_0, we get

$$\langle W_0, V \rangle = \frac{1}{8}(6 + 2 + 2 + 2r - 2(5 - r)) = \frac{4r}{8}.$$

Since this inner product is an integer, we deduce that r is even. For each codeword d, this gives $2^5/2 = 2^4 = 16$ possible vectors. Thus the maximum number of allowed vectors is $21 \cdot 16 = 336$, which we already know is the number of vectors from Case III. Therefore equality holds, and we have specified all the vectors of Case III. Namely, they are all vectors of the form

$$2e_1 + 2e_2 + 2e_3 + 2 \sum_{j \text{ such that } d_j = 1} \pm e_j,$$

where an even number of minus signs are used and d ranges over all codewords in $\mathcal{D}$.

Now we claim that the lattice L is generated by $\sqrt{2}D_{24}$, the vectors in Case III, and W_0, which implies that the vectors in Case IV are uniquely determined. (Recall that they are the only remaining vectors in L that satisfy the constraints enumerated in the paragraph before Equation (1).) To show that L is generated, it suffices to show that the vectors in Case IV are, because all other generators are already included.

For this, let $W = \left(3e_1 + e_2 + e_3 + 2c - \sum_{i=4}^{24} e_i\right)/\sqrt{8}$ be any vector in Case IV. Then $W - W_0 = 2c/\sqrt{8}$ is in the lattice L and it is enough to show that it is in the span of the above generators excluding W_0. Equivalently, we must show that c is in the span of $2(e_i \pm e_j)$ and $e_1 + e_2 + e_3 + d$ with $d \in \mathcal{D}$. Because $2c/\sqrt{8} \in L$ and L is even, the weight of c must be a multiple of 4. Therefore, what we need to show is that in $\mathbb{F}_2^{24}$, the codeword $000c$ is in the span of the codewords $111d$ for $d \in \mathcal{D}$, where of course $000c$ denotes the concatenation of $(0, 0, 0)$ with c. (When we work

modulo 2, the vectors $2(e_i \pm e_j)$ vanish. Fortunately, that is not a problem, because $000c$ and $111d$ all have weights divisible by 4. It follows from congruence modulo 2 that the difference in $\mathbb{Z}^{24}$ between $000c$ and a sum of vectors of the form $111d$ is not only in the span of the vectors $2e_i$ but in fact in the span of $2(e_i \pm e_j)$.)

Conversely, any vector of the form $000c$ that is in the span of the codewords $111d$ for $d \in \mathcal{D}$ will correspond to a vector in Case IV.

Of course one must take the sum of an even number of words $111d$ to arrive at a word of the form $000c$. It is easily checked that the code $\mathcal{D}$ spans a 10-dimensional subspace of $\mathbb{F}_2^{21}$ (simply check that the incidence matrix of the projective plane over $\mathbb{F}_4$ has rank 10 over $\mathbb{F}_2$; this is easily checked directly or by using a general formula that is implicit in [GM] and explicit in [MM] and [S]). Hence the codewords of the form $111d$ with $d \in \mathcal{D}$ span 512 words of the form $000c$. Converting back to vectors, these give us 512 vectors of the form $W - W_0$ with W in Case IV. However, we know that the total number of W's in Case IV is 512. Therefore all of them must come from this construction. In other words, this shows that $000c$ is always in the span of $111d$ with $d \in \mathcal{D}$, which is what we wanted to prove.

This concludes the proof of Theorem 1.

3. Uniqueness of the $(23, 4600, 1/3)$ code

Now we add a correction to the proof in [BS] that there is a unique code of size 4600 and maximum inner product $1/3$ in S^{22}, up to orthogonal transformations of $\mathbb{R}^{23}$. As mentioned above, this code is derived from the Leech lattice by taking the kissing arrangement twice.

Theorem 4. *There is a unique* $(23, 4600, 1/3)$ *spherical code, up to orthogonal transformations of* $\mathbb{R}^{23}$.

Proof. Let $u_1, \dots, u_{4600}$ be the points in the code, and set

$$V_0 = (2, 0, \dots, 0) \in \mathbb{R}^{24}$$

and

$$U_i = (1, \sqrt{3}u_i).$$

Let L be the lattice spanned by V_0 and $U_1, \dots, U_{4600}$.

The analogues of Lemmas 2 and 3 go through as before. However, it is then stated in [BS] that L is the Leech lattice, which is not correct (for by construction, every element of L has even inner product with V_0, which is not true for every vector in the kissing configuration of the Leech lattice). However, one can take the path that we have described above. Briefly, we have the following setup:

Choose new coordinates so that L contains the usual lattice $\sqrt{2}D_{24}$ and $V_0 = (4e_1 + 4e_2)/\sqrt{8}$. The vectors in L that could possibly have inner product 2 with V_0 are of the form $(w_1, \dots, w_{24})/\sqrt{8}$ with

$$(w_1, w_2, \dots, w_{24}) = \begin{cases} 4(e_1 + e_j) & \text{with } j \geq 3, \\ 4(e_2 + e_j) & \text{with } j \geq 3, \\ 2(e_1 + e_2) + 2\sum_{k=1}^{6} \pm e_{j_k} & \text{with } 2 < j_1 < j_2 < \cdots < j_6, \\ 3e_1 + e_2 + \sum_{j=3}^{24} \pm e_j, & \text{or} \\ e_1 + 3e_2 + \sum_{j=3}^{24} \pm e_j. & \end{cases}$$

Call these cases Case I through Case V.

Once again we enumerate the possibilities: Cases I and II lead to 22 vectors each. Case III leads to a $(22, 8, 6)$ code, which has at most 77 elements by the linear programming bounds for constant-weight codes. Therefore there are at most $2^5 \cdot 77 = 2464$ vectors from Case III (as in the previous case only half of the possible sign patterns can occur). Finally, Cases IV and V both lead to $(22, 8)$ codes, so they give at most $2^{10} = 1024$ vectors each by the linear programming bounds for binary codes. The total number of possible vectors is 4600 exactly, i.e., as many as we started with. Therefore the numbers must be exact, and in particular, we can normalize the code $\mathcal{D}$ corresponding to Case III, by the uniqueness of the $(3, 6, 22)$ Steiner system (Satz 4 in [W]). We need to show that the vectors of Case IV and V are determined (up to isometries fixing V_0, $\sqrt{2}D_{24}$, and the code $\mathcal{D}$) from this. Let $W_0 = \left(3e_1 + e_2 - \sum_{i=3}^{24} e_i\right)/\sqrt{8}$ be a vector from Case IV, which we can assume after applying isometries as before. Let V be a vector from Case III with ± 2's at locations in the codeword $d \in \mathcal{D}$, and suppose there are r minus signs and $6 - r$ plus signs. Then

$$\langle W_0, V \rangle = \frac{1}{8}(6 + 2 + 2r - 2(6 - r)) = \frac{-4 + 4r}{8},$$

which forces r to be odd. Now we get $2^6/2$ vectors for each codeword, for 77 codewords. Again exactness shows us that all the vectors of Case III are uniquely determined.

Next we would like to show that vectors of Cases I, II and III span the lattice L, or in particular, the remaining generators from Cases IV and V. It suffices to deal with Case IV since clearly Case V is obtained by subtracting vectors of Case IV from $4(e_1 + e_2)/\sqrt{8} = V_0$.

For Case IV, we employ the same technique used in Case IV for the $(22, 891, 1/4)$ code. It amounts to showing that the linear span of the 77 codewords $11d$ with $d \in \mathcal{D}$ contains exactly 1024 vectors of the form $00c$ with $c \in \mathbb{F}_2^{22}$, which is easily checked on a computer. □

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